Week 9: Homotopy Pushouts II

Tyrone Cutler

July 5, 2020

Contents

1 Introduction

In the exercises this week we will continue to study homotopy pushouts. For the most part this week we will be working with strictly commutative squares and diagrams. To set some conventions and notation we will agree that given a strictly commutative diagram

$$\begin{tikzcd} B \arrow[d, "\beta"] & A \arrow[l, "i"'] \arrow[r, "j"] \arrow[d, "\alpha"] & C \arrow[d, "\gamma"] \\ X & W \arrow[l, "f"] \arrow[r, "g"'] & Y \end{tikzcd} \tag{1.1}$$

the induced map of homotopy pushouts $\theta = \theta(\alpha, \beta, \gamma) : M(i, j) \to M(f, g)$ is given by

$$\theta(b) = \alpha(b), \qquad \theta(a, t) = (\alpha(a), t), \qquad \theta(c) = \gamma(c). \tag{1.2}$$

This is plainly homotopic the map $\theta(F, G)$ introduced last week in the case that F, G are the trivial homotopies, and is a little more convenient to work with.

Please complete all the exercises. There are four in total.

2 Homotopy Colimits of 3×3 Diagrams

Suppose given a strictly commutative diagram

$$\begin{tikzcd}
A_1 & A_0 \arrow[l, "a_1"'] \arrow[r, "a_2"] & A_2 \\
B_1 \arrow[u, "\alpha_1"'] \arrow[d, "\gamma_1"] & B_0 \arrow[l, "b_1"'] \arrow[r, "b_2"] \arrow[u, "\alpha_0"'] \arrow[d, "\gamma_0"] & B_2 \arrow[u, "\alpha_2"'] \arrow[d, "\gamma_2"] \\
C_1 & C_0 \arrow[l, "c_1"'] \arrow[r, "c_2"] & C_2 .
\end{tikzcd} \tag{2.1}$$

which we'll refer to below as $\mathcal{X}$. Taking homotopy pushouts of the rows in the diagram, the vertical arrows induce maps

$$M(a_1, a_2) \xleftarrow{\alpha} M(b_1, b_2) \xrightarrow{\gamma} M(c_1, c_2). \tag{2.2}$$

On the other hand, taking homotopy pushouts of the columns of $\mathcal{X}$, the horizontal arrows induce maps

$$M(\alpha_1, \gamma_1) \xleftarrow{d_1} M(\alpha_0, \gamma_0) \xrightarrow{d_2} M(\alpha_2, \gamma_2). \tag{2.3}$$

Taking homotopy pushouts of either of these two spans results in a space which we might consider to be the *homotopy colimit* of the 3×3 diagram $\mathcal{X}$.

Proposition 2.1 *With the notation above, the double mapping cylinder of the maps in* (2.2) *is homeomorphic to the double mapping cylinder of the maps in* (2.3).

Proof Both spaces are quotients of

$$A_1 \vee (A_0 \wedge I_+) \vee A_2 \vee (B_1 \wedge I_+) \vee (B_0 \wedge I_+ \wedge I_+) \vee (B_2 \wedge I_+) \vee C_1 \vee (C_0 \wedge I_+) \vee C_2 \tag{2.4}$$

by the same set of relations. It is tedious, but not difficult, to check that the order in which we quotient out the necessary relations to generated each of the spaces is inconsequential. $\blacksquare$

We encourage the reader to sketch out some details to convince themselves that everything goes through. Interpreted in another way, the statement says that there are homotopy pushouts

$$\begin{tikzcd}
M(b_1, b_2) \arrow[r] \arrow[d] & M(c_1, c_2) \arrow[d] \\
M(a_1, a_2) \arrow[r] & T(\mathcal{X})
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M(b_1, b_2) \arrow[r] \arrow[d] & M(\alpha_2, \gamma_2) \arrow[d] \\
M(\alpha_1, \gamma_1) \arrow[r] & T(\mathcal{X}).
\end{tikzcd} \tag{2.5}$$

where

$$T(\mathcal{X}) = M(\underline{\alpha}, \underline{\beta}) = M(\underline{d}_1, \underline{d}_2). \tag{2.6}$$

The point is that given the 3×3 diagram $\mathcal{X}$, taking homotopy pushouts of the rows first or the columns first gives rise to the same space. We'll call the space $T(\mathcal{X})$, or more precisely its homotopy type, the **homotopy colimit** of $\mathcal{X}$.

Example 2.1 *The homotopy colimits of the diagrams*

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ * \arrow[u] \arrow[d] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\ * & * \arrow[l] \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] & X \arrow[r, equal] & X \\ X \arrow[u, equal] \arrow[d, equal] & * \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \arrow[u, equal] \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \tag{2.7}$$

are $\Sigma^2 X$ and ΣX, respectively. To check the second homotopy colimit you need to compute the homotopy pushout of $X \xleftarrow{\nabla} X \vee X \xrightarrow{\nabla} X$ (hint: turn just one of the maps into a cofibration).
□

Example 2.2 *Start with the diagram*

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ * \arrow[u] \arrow[d] & W \arrow[l] \arrow[r] \arrow[u] \arrow[d, "f"'] & * \arrow[u] \arrow[d] \\ * & X \arrow[l] \arrow[r] & *. \end{tikzcd} \tag{2.8}$$

Taking homotopy pushouts of the rows we get

$$\Sigma X \xleftarrow{\Sigma f} \Sigma W \to * \tag{2.9}$$

and it follows that the homotopy colimit of (2.8) is the mapping cone $C_{\Sigma f}$. On the other hand, taking homotopy pushouts of the columns in the diagram yields

$$* \leftarrow C_f \to *. \tag{2.10}$$

Thus there is a homotopy equivalence (in this case a homeomorphism)

$$C_{\Sigma f} \simeq \Sigma C_f. \tag{2.11}$$

□

Exercise 2.1 Generalise the outcome of Example (2.2). Show that if

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & Z \end{tikzcd} \tag{2.12}$$

is a homotopy pushout, then so is

$$\begin{tikzcd} \Sigma W \arrow[r, "\Sigma g"] \arrow[d, "\Sigma f"'] & \Sigma Y \arrow[d, "\Sigma k"] \\ \Sigma X \arrow[r, "\Sigma h"] & \Sigma Z \end{tikzcd} \tag{2.13}$$

The hint (if you have not been paying attention) is to construct a 3×3 diagram whose homotopy colimit is both $M(\Sigma f, \Sigma g)$ and $\Sigma M(f, g)$, and identify this space suitably with ΣZ. □

3 Iterated Cofibers

Suppose given a strictly commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & Z \end{tikzcd} \tag{3.1}$$

which we will assume to be equipped with the trivial homotopy. Is this square a homotopy pushout? Well, there is one simple obstruction. The square induces maps

$$\underline{h} : C_f \to C_k, \qquad \underline{k} : C_g \to C_h \tag{3.2}$$

of cofibers, which we saw last week to be homotopy equivalences when the square is a homotopy pushout. If these induced maps are homotopy equivalences, then their cofibers will be contractible. This suggests in the general case that we study the cofibers of these induces maps (3.1).

Exercise 3.1 Use the diagram

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ X \arrow[u] \arrow[d] & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\ Z & Y \arrow[l] \arrow[r] & * \end{tikzcd} \tag{3.3}$$

to show that there is a homotopy equivalence

$$C_{\underline{h}} \simeq C_{\underline{k}}. \tag{3.4}$$

□

The result of the exercise is quite useful, since it reduces the task of studying two spaces to studying just one. We'll call the common homotopy type $C_{\underline{h}} \simeq C_{\underline{k}}$ the **iterated cofiber** of the square (3.1) and denote it suggestively by $C_\square$. The following exercise shows its relevance.

Exercise 3.2 Use the diagram

$$\begin{tikzcd} X \arrow[r, equal] & X \arrow[r] & * \\ X \arrow[u, equal] \arrow[d] & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\ Z & Y \arrow[l] \arrow[r] & * \end{tikzcd} \tag{3.5}$$

to show that there is a cofiber sequence

$$M(f, g) \to Z \to C_\square. \tag{3.6}$$

□

Thus the iterated cofiber of the square is exactly the cofiber of the canonical comparison map $M(f,g) \to Z$.

With a little work we can extend the above discussion to squares which only commute up to homotopy. Let us replace our original square with a homotopy commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & Z \end{tikzcd} \tag{3.7}$$

which we we'll equip with a specific choice of homotopy F so as to make sense of the induced maps. Now factor g as a composite $W \hookrightarrow \overline{Y} \xrightarrow{\simeq} Y$ of a cofibration g' followed by a homotopy equivalence. Since g' is a cofibration we can use the HEP to replace the composite $\overline{Y} \xrightarrow{\simeq} Y \xrightarrow{k} Z$ with a homotopic map $k' : \overline{Y} \to Z$ such that $k'g = hf$ holds strictly. The cofibers of k and k' are then homotopy equivalent, as are the cofibers of g and g' (apply Th. 3.5 from *Homotopy Pushouts I*). Now repeat the previous constructions to the strictly commutative square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f"'] & \overline{Y} \arrow[d, "k'"] \\ X \arrow[r, "h"] & Z. \end{tikzcd} \tag{3.8}$$

Working through the details we get the following omnibus statement.

Proposition 3.1 *Assume given the homotopy commutative diagram* (3.7)*. Then a choice of homotopy F for the square gives rise to the unlabeled maps in a homotopy commutative diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "k"'] & C_g \arrow[d] \\ X \arrow[r, "h"] \arrow[d] & Z \arrow[r] \arrow[d] & C_h \arrow[d] \\ C_f \arrow[r] & C_k \arrow[r] & C_\square \end{tikzcd} \tag{3.9}$$

in which each row and column is a cofiber sequence, and $C_\square$ is the homotopy cofiber of the comparison map $\theta_F : M(f,g) \to Z$. If the square (3.7) *commutes strictly and F is the trivial homotopy, then* (3.9) *commutes strictly.* ∎

Example 3.1 *Suppose maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{3.10}$$

are given. We apply the proposition to the strictly commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "gf"] \\ Y \arrow[r, "g"] & Z \end{tikzcd} \tag{3.11}$$

with the result being a cofiber sequence of the form

$$C_f \to C_{gf} \to C_g. \tag{3.12}$$

□

Example 3.2 *In* Cofiber Sequences II *you were asked to compute the cohomology ring of the space $C(k) = C_{k\cdot\eta}$ obtained as the cofiber of k times the Hopf map $\eta : S^3 \to S^2$. In this case the cofiber sequence* (3.12) *is*

$$P^4(k) \to C(k) \to \mathbb{C}P^2 \tag{3.13}$$

where $P^4(k) = S^3 \cup_k e^4$ is the degree k Moore space. Calculating $H^*C(k)$ is now a simple task of studying the long exact cohomology sequence of (3.13). □

4 Cofibers of Induced Maps

Suppose given a strictly commuting diagram

$$\begin{tikzcd} B \arrow[d, "\beta"] & A \arrow[l, "i"'] \arrow[r, "j"] \arrow[d, "\alpha"] & C \arrow[d, "\gamma"] \\ X & W \arrow[l, "f"] \arrow[r, "g"'] & Y. \end{tikzcd} \tag{4.1}$$

Taking homotopy pushouts of the rows we get a map

$$\theta = \theta(\beta, \alpha, \gamma) : M(i, j) \to M(f, g). \tag{4.2}$$

This map is defined in terms of α, β, γ, and we have seen that when these maps are homotopy equivalences, then so is θ (cf. *Homotopy Pushouts I* Th 3.3). What if α, β, γ are not homotopy equivalences? Can we measure the deviation of θ from being a homotopy equivalence in homotopical terms? Of course the answer again is to study its cofiber.

Exercise 4.1 You're on your own for this one. Construct a commutative 3×3 diagram whose homotopy colimit is C_θ and use it to show that there are homotopy pushouts

$$\begin{tikzcd} M(i,j) \arrow[r] \arrow[d, "\theta"'] & * \arrow[d] \\ M(f,g) \arrow[r] & C_\theta \end{tikzcd} \qquad \begin{tikzcd} C_\alpha \arrow[r] \arrow[d] & C_\gamma \arrow[d] \\ C_\beta \arrow[r] & C_\theta. \end{tikzcd} \tag{4.3}$$

□

This is exactly the result we would hope for. It clearly shows how the transfer of information which is represented by the diagram (4.1) is seen on the level of the homotopy pushouts.

With a bit of work we can make the construction work for diagrams which only commute up to homotopy. Of course we need to be careful to fix particular homotopies and use them throughout. We won't detail this, but rather only give the following slightly imprecise statement.

Proposition 4.1 *Suppose given a homotopy commutative cube*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd, "\alpha"'] & & Y \arrow[dr] \arrow[dd, "\gamma"' near end] & \\
& X \arrow[rr] \arrow[dd, "\beta" near start] & & Z \arrow[dd, "\delta"] \\
W' \arrow[rr] \arrow[dr] & & Y' \arrow[dr] & \\
& X' \arrow[rr] & & Z'
\end{tikzcd} \tag{4.4}$$

in which the top and bottom faces are homotopy pushouts. Then the cofibers of the vertical maps can assemble into a homotopy pushout square

$$\begin{tikzcd}
C_\alpha \arrow[r] \arrow[d] & C_\gamma \arrow[d] \\
C_\beta \arrow[r] & C_\delta.
\end{tikzcd} \tag{4.5}$$

$\blacksquare$

Example 4.1 Let X, Y be spaces. We define their (reduced) **join** $X * Y$ as the double mapping cylinder of the two projections $X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y$. Thus by definition we have a homotopy pushout

$$\begin{tikzcd}
X \times Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"'] & Y \arrow[d] \\
X \arrow[r] & X * Y.
\end{tikzcd} \tag{4.6}$$

Checking directly we see that the maps $X \to X * Y$ and $Y \to X * Y$ in the diagram are null homotopic. Since the diagram is a homotopy pushout this implies, for instance, that the homotopy cofiber of $pr_X : X \times Y \to X$ is equivalent to $X * Y \vee \Sigma Y$. Now consider the strictly commutative diagram

$$\begin{tikzcd}
X \arrow[d, equal] & X \vee Y \arrow[l, "q_X"'] \arrow[r, "q_Y"] \arrow[d] & Y \arrow[d, equal] \\
X & X \times Y \arrow[l, "pr_X"] \arrow[r, "pr_Y"'] & Y
\end{tikzcd} \tag{4.7}$$

where q_X, q_Y are the pinch maps.

Lemma 4.2 *The homotopy pushout of* $X \xleftarrow{q_X} X \vee Y \xrightarrow{q_Y} Y$ *is contractible.*

Proof We check that the double mapping cylinder

$$M(q_X, q_Y) = \frac{X \vee (X \vee Y) \wedge I_+ \vee Y}{\sim} \tag{4.8}$$

is homeomorphic to the contractible space

$$\frac{X \wedge I_+ \vee Y \wedge I_+}{X \wedge \{1\}_+ \vee Y \wedge \{1\}_+} \cong CX \vee CY. \quad \blacksquare \tag{4.9}$$

Now assume that X, Y are well-pointed. Then the inclusion $X \vee Y \hookrightarrow X \times Y$ is a cofibration with cofiber $X \wedge Y$. Thus the cofibers of the vertical maps in (4.7) give a diagram

$$* \leftarrow X \wedge Y \rightarrow * \tag{4.10}$$

whose homotopy pushout is $\Sigma(X \wedge Y)$. On the other hand, the homotopy cofiber of the induced map

$$M(q_X, q_Y) \simeq * \rightarrow X * Y \tag{4.11}$$

is equivalent to $X * Y$ since its domain is contractible. According to Proposition 4.1 these two spaces are homotopy equivalent.

Proposition 4.3 *Let X, Y be well-pointed. Then there is a homotopy equivalence*

$$X * Y \simeq \Sigma(X \wedge Y). \quad \blacksquare \tag{4.12}$$

Putting this with one of our opening observations we get a useful corollary.

Proposition 4.4 *The cofiber of the projection $pr_X : X \times Y \rightarrow X$ is homotopy equivalent to $\Sigma(X \wedge Y) \vee \Sigma Y$.* $\blacksquare$

Remark The join is an important space both geometrically and homotopically. You may enjoy computing its homology using the tools given in the last example. There is similarly an *unreduced* join $X \widetilde{*} Y$, defined as the unreduced homotopy pushout of the projections $X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y$ in the unpointed category. If X, Y are well-pointed, then the quotient maps $X \widetilde{*} Y \rightarrow X * Y \rightarrow \Sigma(X \wedge Y)$ are both homotopy equivalences. $\square$

4.1 The Mayer-Vietoris Sequence

Let

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & Z \end{array} \tag{4.13}$$

be a homotopy pushout. We make use of the following diagram

$$\begin{array}{ccccc} X & \longleftarrow & * & \longrightarrow & Y \\ \| & & \downarrow & & \| \\ X & \xleftarrow{f} & W & \xrightarrow{g} & Y. \end{array} \tag{4.14}$$

to induce a map of homotopy pushouts, the result being the obvious map

$$(h, k) : X \vee Y \rightarrow Z. \tag{4.15}$$

On the other hand, according to Proposition 4.1 the cofiber of this map is the homotopy pushout of

$$* \leftarrow W \rightarrow *. \tag{4.16}$$

Of course this space is the suspension ΣW. Thus we can find a map $\delta : Z \to \Sigma W$ such that

$$X \vee Y \xrightarrow{(h,k)} Z \xrightarrow{\delta} \Sigma W \tag{4.17}$$

is a cofiber sequence. With a little work we can show that this sequence extends as

$$X \vee Y \xrightarrow{(h,k)} Z \xrightarrow{\delta} \Sigma W \xrightarrow{\Sigma f - \Sigma g} \Sigma X \vee \Sigma Y \xrightarrow{\Sigma h + \Sigma k} \Sigma Z \to \dots . \tag{4.18}$$

We call (4.18) the **Mayer-Vietoris sequence** of the homotopy pushout (4.13).

To check that the maps which we have claimed above are correct let us assume that (4.13) is the standard homotopy pushout of f, g. Then $Z = M(f, g)$ can be realised as the topological pushout

$$\begin{tikzcd} W \vee W \arrow[r, "f \vee g"] \arrow[d, "in_0 \vee in_1"'] \arrow[dr, phantom, "\lrcorner"] & X \vee Y \arrow[d, "(h,k)"] \\ W \wedge I_+ \arrow[r] & M(f, g). \end{tikzcd} \tag{4.19}$$

The left-hand map is the coproduct of the inclusions into the top and bottom of the cylinder, and this map is a cofibration. It follows that the pushout map (h, k) is a cofibration, and in partickular that its homotopy cofiber coincides with its strict cofiber, which identifies canonically with $\Sigma W \cong W \wedge I_+/(W \vee W)$. Thus the map δ introduced above is the obvious collapse map

$$\delta : M(f, g) \to \Sigma W \tag{4.20}$$

which pinches the ends of the mapping cylinders to a point. Really this is obvious in light of (4.14).

Now extend the vertical arrows in (4.19) downwards into *strict* cofiber sequences

$$\begin{tikzcd} W \wedge I_+ \arrow[r] \arrow[d] & M(f, g) \arrow[d] \\ \Sigma W \arrow[r, equal] \arrow[d] & \Sigma W \arrow[d, "\varphi"] \\ \Sigma W \vee \Sigma W \arrow[r, "\Sigma f \vee \Sigma g"] \arrow[d, "-\Sigma in_0 \vee -\Sigma in_1"'] & \Sigma X \vee \Sigma Y \arrow[d, "(\Sigma h, \Sigma k)"] \\ \vdots & \vdots \end{tikzcd} \tag{4.21}$$

We see that the connecting map in which we are interested, here labelled φ, is the composite

$$\Sigma W \to \Sigma W \vee \Sigma W \xrightarrow{\Sigma f \vee \Sigma g} \Sigma X \vee \Sigma Y \tag{4.22}$$

where the first map is the connecting map in the left-hand cofiber sequence. It's not difficult to see that this unlabeled map is the sum

$$\Sigma in_1 - \Sigma in_2 : \Sigma W \to \Sigma W \vee \Sigma W \tag{4.23}$$

so in particular

$$\varphi = \Sigma f - \Sigma g \tag{4.24}$$

as claimed.

5 A Note of Caution

In this sheet we have frequently suggested to use the cofiber of a map $f : X \to Y$ to measure its deviation from being homotopy equivalence. This suggestion is to be taken with care, as it is not always the case that $C_f \simeq *$ implies that f is an equivalence. For example consider the following. The *Poincaré homology sphere* $\mathcal{P}$ is a closed three-dimensional manifold which can be constructed as a quotient of S^3. The fundamental group of $\mathcal{P}$ is a finite group of order 120 known as the *binary icosahedral group*. The key feature of the group for this example is that it is *perfect*, meaning that it is equal to its commutator subgroup. In particular, its abelianisation is trivial. It is also known that $\widetilde{H}^*\mathcal{P} \cong \widetilde{H}^*S^3$.

Now let $X = \mathcal{P} \setminus \{p\}$ be the result of removing a point from the Poincaré homology sphere. Then we can compute

$$\pi_1 X \cong \pi_1 \mathcal{P}, \qquad \widetilde{H}^* X = 0. \tag{5.1}$$

The first of these equations implies that $X \not\simeq *$. On the other hand, since X is connected and well-pointed, the Seifert-van Kampen Theorem implies that $\pi_1 \Sigma X = 0$, and with this we are free to apply the Hurewicz Theorem to conclude that

$$\pi_2 \Sigma X \cong H_2 \Sigma X \cong H_1 X \cong (\pi_1 X)_{Ab} = 0. \tag{5.2}$$

Consequently X is a simply connected space of CW homotopy type which has the homology of a point. One implication of this is that

$$\Sigma X \simeq *. \tag{5.3}$$

Proposition 5.1 *There exist noncontractible spaces with contractible suspensions.* ∎

For another such space see Hatcher's example 2.38 on pg. 142 of *Algebraic Topology*. This space is the cofiber of the map $\varphi : S^1 \vee S^1 \to S^1 \vee S^1$ which is given by $\varphi = (x^5 y^{-1}, x^3(xy)^{-2})$ in terms of the canonical generators $x = in_1, y = in_2 \in \pi_1(S^1 \vee S^1)$.

Back to the point, if X is as in 5.1, then

$$X \to * \to * \tag{5.4}$$

is a homotopy cofiber sequence, but $X \to *$ is not a homotopy equivalence. Thus we have an example of a map which is not a homotopy equivalence, but whose mapping cone is contractible. We also have an example of a homotopy pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & * \arrow[d, "k"] \\ * \arrow[r] & * \end{tikzcd} \tag{5.5}$$

in which k is a homotopy equivalence, but f is not (compare *Homotopy Pushouts I* Pr 4.1). The example we will end with, though, is that of the commutative square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & *. \end{tikzcd} \tag{5.6}$$

This square is *not* a homotopy pushout, but its iterated cofiber is contractible.

References

[1] F. Borceux, *Handbook of Categorical Algebra I: Basic Category Theory*, Cambridge University Press, 1994.

[2] R. Fritsch, R. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[3] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.